DFQ - STMATH 307

- equations involving a function and its derivatives
- deals with rates of change
- mathematical model → simplified

In general, for $y' = ay - b$, find eq. soln by setting $y' = 0$

$y(t) = \frac{b}{a}$ - can have several eq. solutions

steps to construct a DFQ

- identify independent + dependent variables
- assign them names
- make units cooperate

Monday, 30 Sep '13, DQQ Notes 9/30

We found the general solution for $p' = .5p - 450$

$$\frac{dP}{dt} = .5(p - 900) \Rightarrow \frac{dP/dt}{p-900} = .5 \Rightarrow \int \frac{dP}{P-900} = \int .5\,dt \Rightarrow \ln|p-900| = .5t + C$$

$$\rightarrow |p - 900| = e^{(.5t + C)} = e^{.5t} e^{C} = ke^{.5t} \Rightarrow \underline{P = 900 + ke^{.5t}}$$

$k = \pm e^{C}$

Prove that P is a solution.

$$\frac{dP}{dt} = \frac{k}{2} \cdot e^{(t/2)} \rightarrow \text{plug in} \quad \frac{k}{2} e^{t/2} = .5p - 450$$

$$.5(900 + ke^{.5t}) - 450$$

$$= \frac{k}{2} e^{.5t} \; \square$$

Purple Freakin' Differential Equations Notes (9/30)

- An initial condition determines a unique solution (solving for C)
  e.g. $y(0)=0$; $p(0)=850$; $p(2)=4$

a general solution has an arbitrary constant.

Solving $y'=ay-b \to \frac{dy}{dt}=a\left(y-\frac{b}{a}\right) \to \underline{y=\frac{b}{a}+ke^{at}}$ — for a positive, y increases; for a negative, y goes to b/a exp. decay

Solving for equilibrium solution, set y' to $\emptyset$ and solve for y.

$\frac{dv}{dt}=9.8-.2v$ suppose an object is dropped from 300 m. above ground find v(t) and velocity at impact.

$\frac{dv}{dt}=-.2(v-49) \to \int \frac{dv}{(v-49)} = \int -.2\,dt \to \ln(|v-49|)=-.2t+C$

$v-49=e^{-.2t+C}$

$v=49-e^{-.2t}e^{C}$

★ check this ★ Should have got $V(t)=49(1-e^{-.2t})$

$S(t)=\int v(t) \to s'(t)=v(t) \to$ solve for $S(t)=300$ for t

$s'(t)=v(t)=49-49e^{-.2t} \to s(t)=49t+245e^{-.2t}+C;\ s(0)=0 \to C=-245$

$S(t)=49t+245e^{-.2t}-245=300.\ t\approx 10.51,\ v\approx 43.01$ m/sec

Ordinary Differential Equations are based on single variable functions

Partial Differential Equations involve multivariable functions and $\frac{\partial x}{\partial t}$ partials

We can solve

① $y^{(n)}=f\left(t,y,y',y'',\ldots,y^{(n-1)}\right).$

Linear O.D.E.'s look like

$a_0(t)y^{(n)}+a_1(t)y^{(n-1)}+\cdots+a_n(t)y=g(t)$ ②

A solution $\phi(t)$ to an O.D.E like ① is like ③

Solutions. A **solution** of the ordinary differential equation (8) on the interval $\alpha<t<\beta$ is a function ϕ such that $\phi',\phi'',\ldots,\phi^{(n)}$ exist and satisfy ③

$$\phi^{(n)}(t)=f\left[t,\phi(t),\phi'(t),\ldots,\phi^{(n-1)}(t)\right]$$

Is there a solution? If so, is it unique? If so, how do we find it?

Skip Lester 30 Sep 2013
Differential Equations

1.1

21a) A pond initially contains 1,000,000 gallons of water and an unknown amount of an undesirable chemical. Water containing 0.04 gram of this chemical per gallon flows into the pond at a rate of 200 gal/hour. The mixture flows out at the same rate, so the amount of water in the pond remains constant. Assume the chemical is uniformly distributed in the pond. Write a Differential Equation for the amount q (in grams) of chemical in the pond at any time t (in hours).

$$\frac{dq}{dt} = 8 - \frac{200q}{1{,}000{,}000}$$

21b) How many grams will be left after a very long time?

$$\frac{dq}{dt} = (0.04)(200) - \frac{200\,q}{1{,}000{,}000} \quad \text{- set to zero}$$

$$0 = 8 - \frac{2}{10{,}000}q$$

$$5{,}000\left(\frac{2}{10{,}000}\right)q = 8(5{,}000)$$

$q = \underline{40{,}000 \text{ grams}}$ after large T

24a) A drug is being administered intravenously to a hospital patient. Fluid containing $k = 7\,mg/cm^3$ of the drug enters the patients bloodstream at a rate of $r = 50\,cm^3/hour$. The drug is absorbed by body tissue or otherwise leaves the bloodstream at a rate proportional to the amount present, with a rate constant of $p = 0.10\,(hr^{-1})$. Write a differential equation for the amount of drug present in the bloodstream at any time t.

$$\frac{dq}{dt} = k\cdot r - pq \rightarrow \frac{dq}{dt} = (7\,mg/cm^3)(50\,cm^3/hr) - (0.1\,1/hr)(q(t))$$

$$= \underline{350\,mg/hr - .1q} = 0$$

How much of the drug is present after a long time?

$.1q = 350$

$q = 3500$

Skip Lester

STMath 307 Autumn 2013

Worksheet

Solutions to differential equations

$f'(x)g(x)+g'(x)f(x)$ $f(x)=p(t)$ $g(x)=(7-P(t))$

1. Suppose that the function $P(t)$ satisfies the differential equation $P'(t)=P(t)(7-P(t))$ with the initial condition $P(0)=1$. Find $P''(0)$.

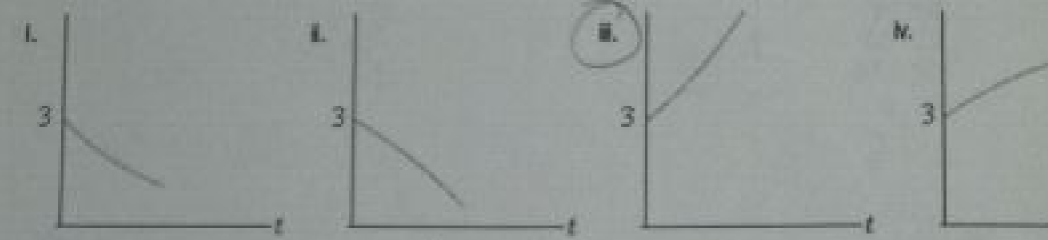
$=7p(t)-[p(t)]^2$

$p'(t)(7-P(t))+(-p'(t))p$

$[7p'(t)-p(t)p'(t)]+[-p'(t)p(t)]$

$7p'-2pp'=p'(7-2p)$

2. Suppose that the function $P(t)$ satisfies the differential equation $P'(t)=P(t)(8-P(t))$ with the initial condition $P(0)=3$. Which of the following is a possible graph for $P(t)$ for small $t>0$?



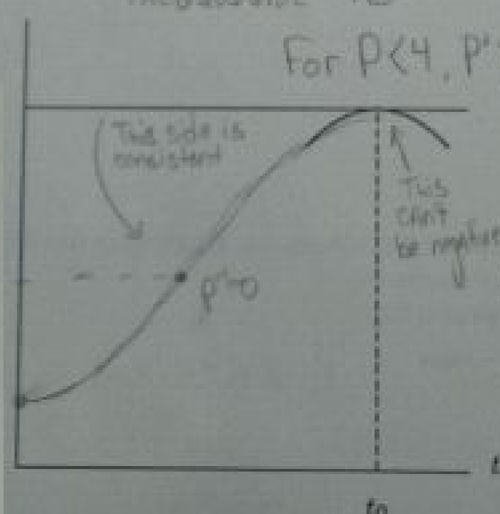
$p''=p'(8-2p)$

3. Suppose that the function $P(t)$ satisfies the differential equation $P'(t)=P(t)(4-P(t))$ with the initial condition $P(0)=1$. Consider the behavior of the graph of $P(t)$ near a point t_0, where $P(t_0)=4$ (if such a point exists). Is the following graph consistent with $P(t)$?

incosistent b/c

For $P<4$, $P'>0$

$4p-p^2$



$P'(0)=1(4-1)=3$

$p'(t)=4p-p^2$

$p''(t)=4p'-2pp'$

$p''(t)=p'(4-2p)$

$p''(0)=3(4-2)$

$=6$

$p'(t_0)=0$

$-2\cdot 2\sin(t\cdot 2)$ $-4\cos(2t)+2(2\cos)2t$

4. Does $y=2\cos(2t)$ satisfy $\frac{d^2y}{dt^2}+2y=0$? No

5. Select the value(s) of w for which $y=e^{wt}$ satisfies $\frac{d^2y}{dt^2}-36y=0$.

A) 8
B) -8
C) 64
D) -64
E) 0
F) 6
G) -6

± 6

we^{wt}

$w^2e^{wt}-36e^{wt}=0$

$e^{wt}(w^2-36)=0$

$(w^2-36)=0$

6. Which of the following functions are solutions to the differential equation $\frac{dy}{dx}=\frac{y}{6}$? (Mark all correct answers.)

A) $y=\sin(6x)+\cos(6x)$
B) $y=6\cos x-\sin x$
C) $y=e^{x/6}$
D) $y=e^{-x/6}$

$\forall x, \int e^x \mathrm{dx} = e^x$

7. Which of the following functions are solutions to the differential equation $\frac{d^2y}{dx^2}=-y$? (Mark all correct answers)

A) $y=\sin(7x)+\cos(7x)$
B) $y=7\cos x-\sin x$ $-7\sin x-\cos x$
C) $y=e^{x/7}$
D) $y=e^{-x/7}$

8. Is $y=e^x\sin x$ a solution to $\frac{d^2y}{dx^2}-\frac{dy}{dx}+y=0$?

$y'=e^x\sin x+e^x\cos x$

$y''=e^x\sin x+e^x\cos x+e^x\cos x-e^x\sin x$

$=2e^x\cos x$

$0=2e^x\cos x-e^x\sin x+e^x\cos x+e^x\sin x$

$0\neq 3e^x\cos x$

Linear first order with variable coefficients   Integrating Factors

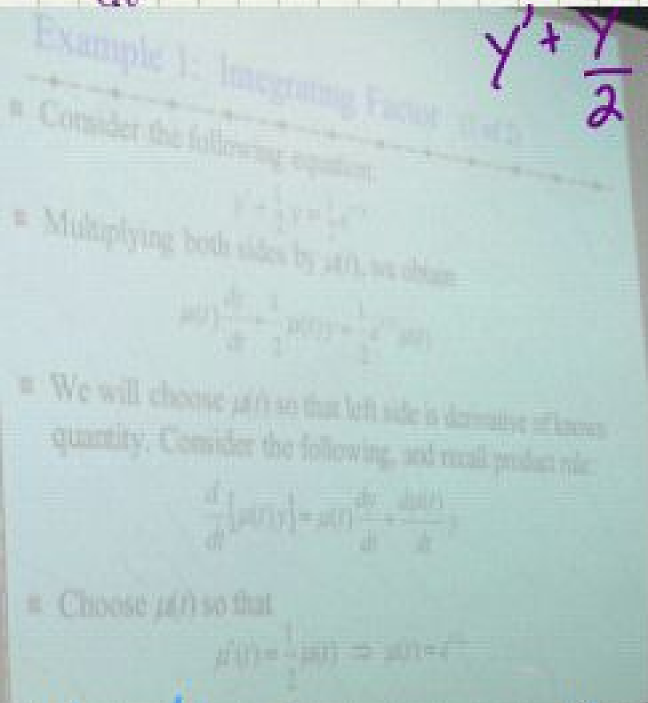
$$\frac{dy}{dt} + p(t)y = g(t)$$

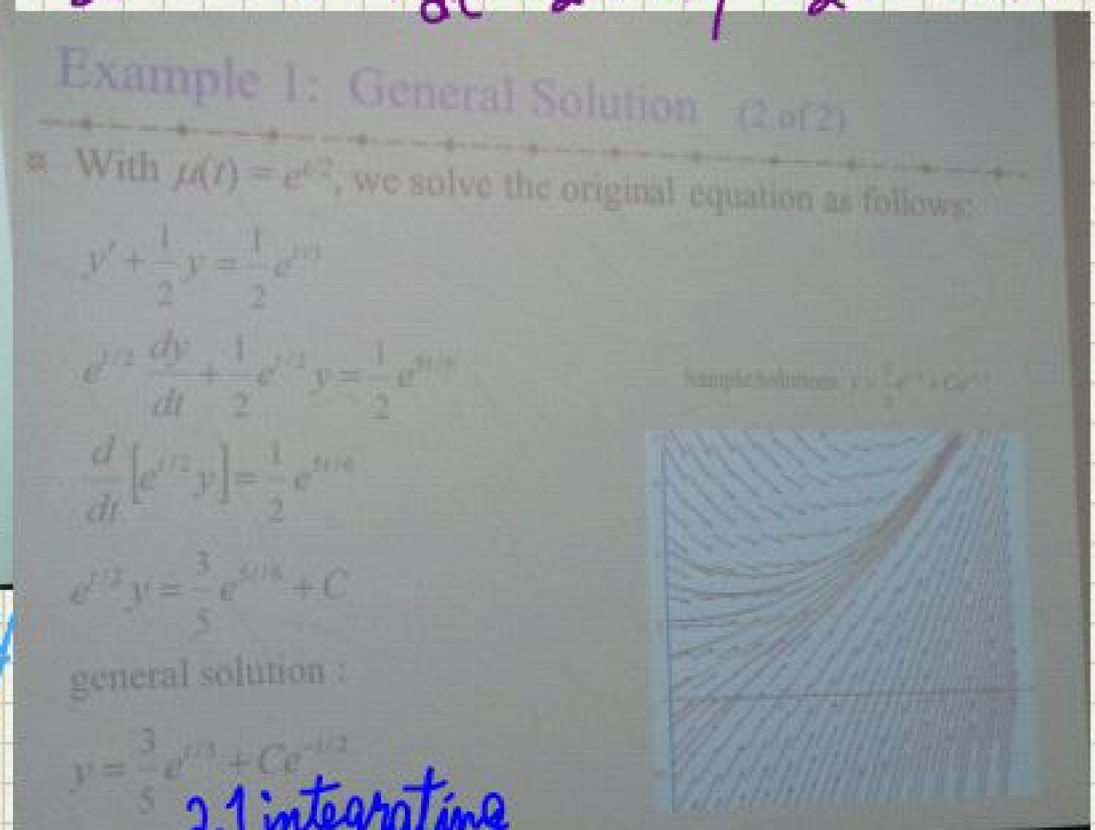
Consider

$$y' + \frac{y}{2} = \frac{e^{t/3}}{2}$$

multiplying by $\mu(t)$, we obtain

$$\mu(t)\frac{dy}{dt} + \frac{1}{2}\mu(t)y = \frac{1}{2}e^{t/3}\mu(t)$$





short quiz on Monday

In general,

$$y' + ay = g(t)$$

$$\mu(t)y' + a\mu(t)y = \mu(t)g(t)$$

$$e^{at}\frac{dy}{dt} + ae^{at}y = e^{at}g(t)$$

$$\frac{d}{dt}\left[e^{at}y\right] = e^{at}g(t) \qquad y' - 2y = 4 - t$$

$$e^{at}y = \int e^{at}g(t)\,dt + C$$

$$y = e^{-at}\int e^{at}g(t)\,dt + Ce^{-at}$$

$$y = e^{2t}\int e^{-2t}(4-t)\,dt + Ce^{2t}$$

$$e^{2t}\int 4e^{-2t} - te^{-2t}\,dt + Ce^{2t}$$

$$e^{2t}\left(-2e^{-2t} - \int te^{-2t}\,dt + Ce^{2t}\right.$$

$$e^{2t}\left(-2e^{-2t} - \left[-\frac{1}{2}te^{-2t} + \int \frac{1}{2}e^{-2t}\,dt\right]\right)$$

$$= e^{2t}\left(-\frac{7}{4}e^{-2t} + \frac{1}{2}te^{-2t}\right) + Ce^{2t}$$

2.1 integrating

To solve

$$ty' + 2y = 4t^2 \quad y(1) = 2$$

$$y' + \frac{2y}{t} = 4t, \; t \neq \emptyset$$

$$\mu(t) = e^{\int p(t)dt} = e^{\int \frac{2}{t}dt} = e^{2\ln(t)} = e^{\ln(t^2)} = t^2$$

$$y = \frac{\int \mu(t)g(t)\,dt + C}{\mu(t)} = \frac{\int t^2(4t)\,dt}{t^2} = \frac{1}{t^2}\left[\int 4t^3\,dt + C\right]$$

$$y = t^2 + C/t^2$$

$$y' + p(t)y = g(t)$$

$$\frac{dy}{dt} + p(t)y = g(t),$$

solve $2y' + ty = 2$, $y(0) = 1$

$$\frac{2y'}{2} + \frac{ty}{2} = \frac{2}{2} \rightarrow y' + \frac{ty}{2} = 1$$

$$\mu = e^{\int \frac{t}{2}y\,dt} = e^{(t^2/4)}$$

Formula for Integration by Parts

Let $u = f(x)$ and $v = g(x)$

The differentials are

$du = f'(x)\,dx$ and $dv = g'(x)\,dx$

By the Substitution Rule,

$$\int u\,dv = uv - \int v\,du$$

Sometimes you may need to integrate by Parts twice!

Integrate $\int e^{-t^2/4} dt \rightarrow$ must approximate

$$y' + 2y = te^{-2t}, \; y(1) = 0$$

- already in standard form $\rightarrow P(t) = 2 \rightarrow \mu(t) = e^{\int 2dt} \rightarrow e^{2t}$

• multiply both sides by $\mu(t)$

$$y'e^{2t} + 2ye^{2t} = te^{-2t}(e^{2t}) \rightarrow \int \frac{d}{dt}\left[e^{2t} y\right] = \int t \, dt$$

$$\rightarrow e^{2t} y = \frac{t^2}{2} + C$$

$$y = e^{-2t}\left[\frac{t^2}{2} + C\right]$$

or $\frac{t^2 e^{-2t}}{2} + ce^{-2t} = y$

$$y(1) = 0 \leftrightarrow \frac{1^2 e^{-2}}{2} + ce^{-2(1)} = 0$$

$$e^{-2}\left(\frac{1}{2} + C\right) = 0$$

$$C = -\frac{1}{2} \rightarrow y(t) = \frac{t^2 e^{-2t}}{2} - \frac{e^{-2t}}{2}$$

$$y(t) = \frac{1}{2} e^{-2t}(t^2 - 1)$$

Chapter 2.2 Seperable Equations

Consider $\frac{dy}{dx} = f(x,y)$; We can re-write this as $M(x,y) + N(x,y)\frac{dy}{dx} = 0$

Then $M(x)dx + N(y)dy = 0$

- can result in implicit relation of x and y.

seperable can be linear or non linear

ex] $\frac{dy}{dx} = \frac{x^2}{1-y^2} \rightarrow (1-y^2)dy = x^2 dx$

$$y - \frac{1}{3}y^3 = \frac{1}{3}x^3 + C$$

2.2 Separable Equations

In Section 1.2 and Section 2.1 we used a process of direct integration to solve first order linear equations of the form

$$\frac{dy}{dt} = ay + b, \tag{1}$$

where a and b are constants. We will now show that this process is actually applicable to a much larger class of equations.

We will use x, rather than t, to denote the independent variable in this section for two reasons. In the first place, different letters are frequently used for the variables in a differential equation, and you should not become too accustomed to using a single pair. In particular, x often occurs as the independent variable. Further, we want to reserve t for another purpose later in the section.

The general first order equation is

$$\frac{dy}{dx} = f(x, y). \tag{2}$$

Linear equations were considered in the preceding section, but if Eq. (2) is nonlinear, then there is no universally applicable method for solving the equation. Here, we consider a subclass of first order equations that can be solved by direct integration.

To identify this class of equations, we first rewrite Eq. (2) in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0. \tag{3}$$

It is always possible to do this by setting $M(x, y) = -f(x, y)$ and $N(x, y) = 1$, but there may be other ways as well. If it happens that M is a function of x only and N is a function of y only, then Eq. (3) becomes

$$M(x) + N(y)\frac{dy}{dx} = 0. \tag{4}$$

Such an equation is said to be **separable**, because if it is written in the differential form

$$M(x)dx + N(y)dy = 0, \tag{5}$$

then, if you wish, terms involving each variable may be placed on opposite sides of the equation. The differential form (5) is also more symmetric and tends to suppress the distinction between independent and dependent variables.

A separable equation can be solved by integrating the functions M and N. We illustrate the process by an example and then discuss it in general for Eq. (4).

12:08 PM

7.3 Separable Equations

Some differential equations can be solved explicitly. A **separable equation** is a first order differential equation in which the expression for $\frac{dy}{dx}$ can be factored as a function of x multiplied by a function of y.

$$\frac{dy}{dx} = g(x)\, f(y)$$

If $f(y) \neq 0$, we can write $\frac{dy}{dx} = \frac{g(x)}{h(y)},\ h(y) = \frac{1}{f(y)}$.

Now we write it with x on one side and y on the other

$h(y)\,dy = g(x)\,dx$...so we can integrate both sides! $\int h(y)\,dy = \int g(x)\,dx$

Sometime we can even solve for y in terms of x:

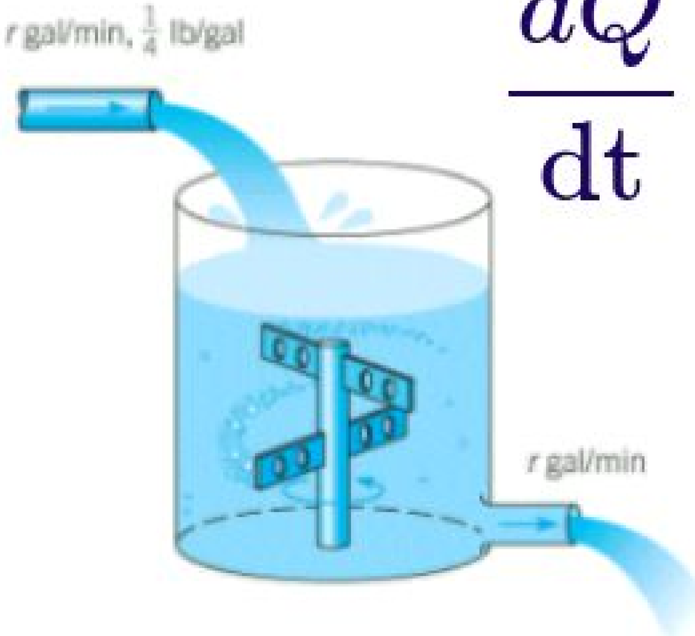
Differentiating implicitly on the left hand side and explicitly on the right,

$$\frac{d}{dx}\left(\int h(y)\,dy\right) = \frac{d}{dx}\left(\int g(x)\,dx\right) \therefore \frac{d}{dy}\left(\int h(y)\,dy\right)\frac{dy}{dx} = g(x)$$

$$\therefore h(y)\frac{dy}{dx} = g(x)$$

Mixing

At time $t = 0$ a tank contains Q_0 lb of salt dissolved in 100 gal of water; see Figure 2.3.1. Assume that water containing $\frac{1}{4}$ lb of salt/gal is entering the tank at a rate of r gal / min and that the well-stirred mixture is draining from the tank at the same rate. Set up the initial value problem that describes this flow process. Find the amount of salt $Q(t)$ in the tank at any time, and also find the limiting amount Q_L that is present after a very long time. If $r = 3$ and $Q_0 = 2Q_L$, find the time T after which the salt level is within 2% of Q_L. Also find the flow rate that is required if the value of T is not to exceed 45 min.



$$\frac{dQ}{dt} = (\text{rate in}) - (\text{rate out})$$

FIGURE 2.3.1 The water tank in Example 1.

We assume that salt is neither created nor destroyed in the tank. Therefore, variations in the amount of salt are due solely to the flows in and out of the tank. More precisely, the rate of change of salt in the tank, dQ / dt, is equal to the rate at which salt is flowing in minus the rate at which it is flowing out. In symbols,

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}. \tag{1}$$

The rate at which salt enters the tank is the concentration $\frac{1}{4}$ lb / gal times the flow rate r gal / min, or $(r / 4)$ lb / min. To find the rate at which salt leaves the tank, we need to multiply the concentration of salt in the tank by the rate of outflow, r gal / min. Since the rates of flow in and out are equal, the volume of water in the tank remains constant at 100 gal, and since the mixture is "well-stirred," the concentration throughout the tank is the same, namely, $[Q(t) / 100]$ lb / gal. Therefore, the rate at which salt leaves the tank is $[rQ(t) / 100]$ lb / min. Thus the differential equation governing this process is

$$\frac{dQ}{dt} = \frac{r}{4} - \frac{rQ}{100}. \qquad Q' + \frac{r}{100}Q = \frac{r}{4} \qquad \mu(t) = e^{\frac{r}{100}\cdot t} \tag{2}$$

The initial condition is

$$Q(0) = Q_0. \tag{3}$$

Upon thinking about the problem physically, we might anticipate that eventually the mixture originally in the tank will be essentially replaced by the mixture flowing in, whose concentration is $\frac{1}{4}$ lb / gal. Consequently, we might expect that ultimately the amount of salt in the tank would be very close to 25 lb. We can also find the limiting amount $Q_L = 25$ by setting dQ / dt equal to zero in Eq. (2) and solving the resulting algebraic equation for Q.

To solve the initial value problem (2), (3) analytically, note that Eq. (2) is both linear and separable. Rewriting it in the standard form for a linear equation, we have

$$e^{\frac{rt}{100}}\left[\frac{dQ}{dt} + \frac{rQ}{100}\right] = \frac{r}{4}e^{\frac{rt}{100}} \rightarrow \int \frac{d}{dt}\left[e^{\frac{rt}{100}}Q\right]dt = \int \frac{re^{\frac{rt}{100}}}{4}\,dt \tag{4}$$

Thus the integrating factor is $e^{rt/100}$ and the general solution is

$$Q(t) = 25 + ce^{-rt/100}. \qquad Q = e^{\frac{-rt}{100}}\left[\int \frac{re^{\frac{rt}{100}}}{4}\,dt\right] \tag{5}$$

where c is an arbitrary constant. To satisfy the initial condition (3), we must choose $c = Q_0 - 25$. Therefore, the solution of the initial value problem (2), (3) is

$$Q(t) = 25 + (Q_0 - 25)e^{-rt/100}, \tag{6}$$

Continued →

or

$$Q(t) = 25(1 - e^{-rt/100}) + Q_0e^{-rt/100} \tag{7}$$

(continued)

$$Q(t) = 25 + (Q_0 - 25)e^{-rt/100}, \quad (6)$$

or

$$Q(t) = 25\left(1 - e^{-rt/100}\right) + Q_0 e^{-rt/100}. \quad (7)$$

From Eq. (6) or (7), you can see that $Q(t) \to 25(\text{lb})$ as $t \to \infty$, so the limiting value Q_L is 25, confirming our physical intuition. Further, $Q(t)$ approaches the limit more rapidly as r increases. In interpreting the solution (7), note that the second term on the right side is the portion of the original salt that remains at time t, while the first term gives the amount of salt in the tank as a consequence of the flow processes. Plots of the solution for $r = 3$ and for several values of Q_0 are shown in Figure 2.3.2.

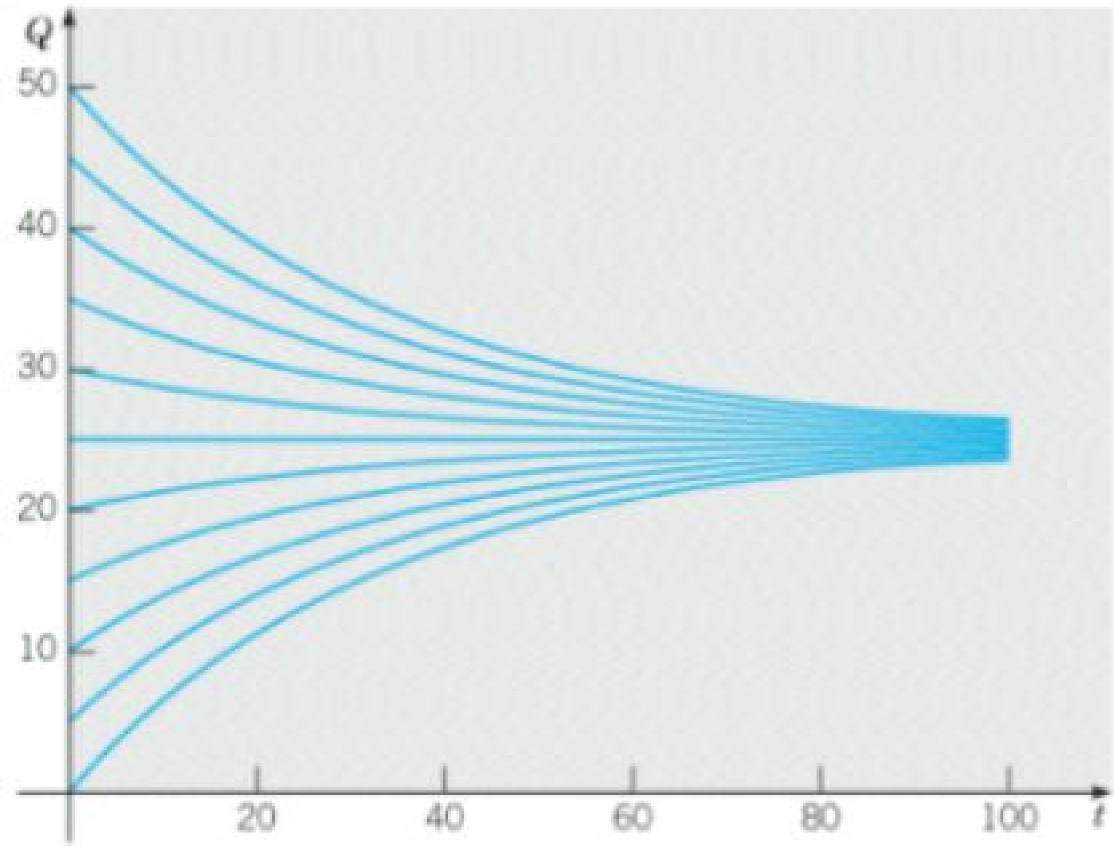


FIGURE 2.3.2 Solutions of the initial value problem (2), (3): $dQ/dt = (r/4) - rQ/100$, $Q(0) = Q_0$ for $r = 3$ and several values of Q_0.

Now suppose that $r = 3$ and $Q_0 = 2Q_L = 50$; then Eq. (6) becomes

$$Q(t) = 25 + 25e^{-0.03t}. \quad (8)$$

Since 2% of 25 is 0.5, we wish to find the time T at which $Q(t)$ has the value 25.5. Substituting $t = T$ and $Q = 25.5$ in Eq. (8) and solving for T, we obtain

$$T = (\ln 50)/0.03 \cong 130.4(\text{min}). \quad (9)$$

To determine r so that $T = 45$, return to Eq. (6), set $t = 45$, $Q_0 = 50$, $Q(t) = 25.5$, and solve for r. The result is

$$r = (100/45)\ln 50 \cong 8.69 \text{ gal/min}. \quad (10)$$

Since this example is hypothetical, the validity of the model is not in question. If the flow rates are as stated, and if the concentration of salt in the tank is uniform, then the differential equation (1) is an accurate description of the flow process. Although this particular example has no special significance, models of this kind are often used in problems involving a pollutant in a lake, or a drug in an organ of the body, for example, rather than a tank of salt water. In such cases the flow rates may not be easy to determine or may vary with time. Similarly, the concentration may be far from uniform in some cases. Finally, the rates of inflow and outflow may be different, which means that the variation of the amount of liquid in the problem must also be taken into account.

$\frac{dy}{dx} = \frac{x^4}{y} \leftrightarrow \int y\,dy = \int x^4\,dx \rightarrow \frac{y^2}{2} - \frac{x^5}{5}$

$\leftrightarrow \frac{y^2}{2} = \frac{x^5}{5} + C \qquad \frac{2y^2}{2} - \frac{2x^5}{5}$

$y = \pm\sqrt{2\left(\frac{x^5}{5} + C\right)} \qquad 5y^2 - 2x^5$

$\int 11 + 3y\,dy = \int 11x^2 - 1\,dx$

$11y + \frac{3y^2}{2} = \frac{11}{3}x^3 - x + C$

$\frac{GMm}{R^2}$ ← Gravity

$\int v\,dv = \int \frac{-gR^2}{(R+x)^2}\,dx$

$\frac{KQq}{R^2}$ ← Electricity

$\frac{v^2}{2} = +gR^2/(R+x) + C$

$\frac{dy}{(1-y/k)y} = r\,dt \rightarrow \int \frac{1}{\left(y^2 - \frac{y}{k}\right)}\,dy = \int r\,dt \qquad \frac{1}{\left(1-\frac{y}{k}\right)y} = \frac{A}{1-y/k} + \frac{B}{y} \rightarrow$

$1 = Ay + B(1 - y/k) = Ay + B - \frac{By}{k}$

$A = \frac{1}{k} \quad B = 1$

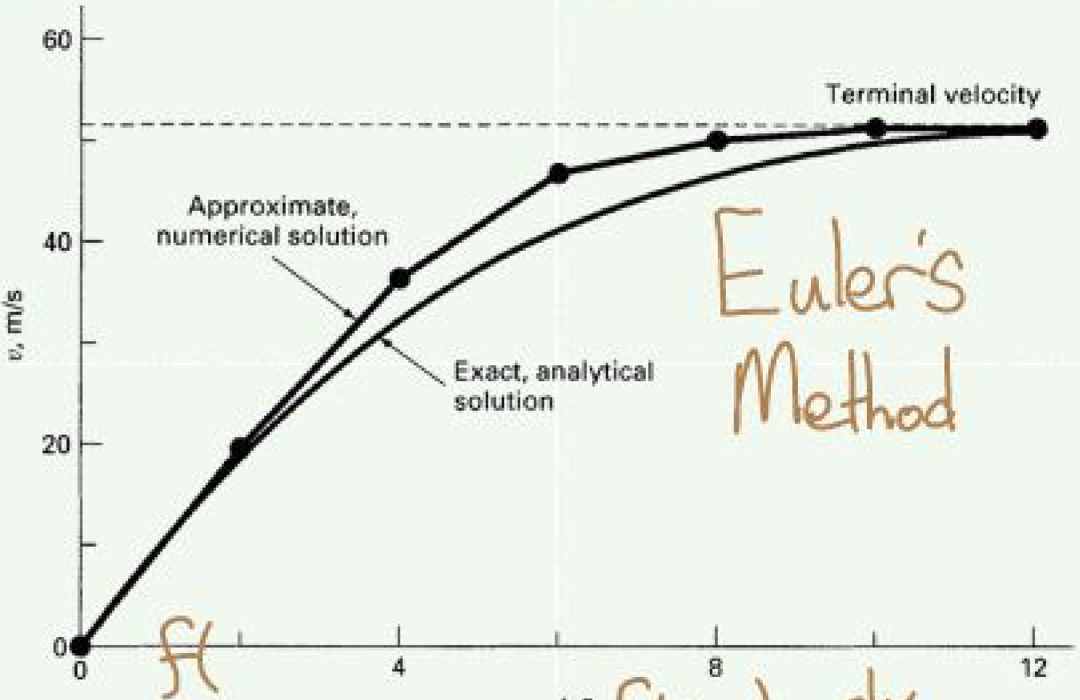
$\frac{y}{k} + 1 - \frac{y}{k} = 1 \; \square$

$\int \frac{dx}{(x-3)(x-4)} \Rightarrow \frac{1}{(x-3)(x-4)} = \frac{A}{(x-3)} + \frac{B}{(x-4)}$

Logistic Equation

$1 = A(x-4) + B(x-3)$

$A = -1, \; B = 1$



Euler's Method

f(

$f(t,y) = \frac{dy}{dt}$

$y_1 = y_0 + f_0(t,y)\cdot h \quad h = \Delta t$

The relative error can also be multiplied by 100% to express it as

$$\varepsilon_t = \frac{\text{true value} - \text{approximation}}{\text{true value}} 100\%$$

$y_2 = 1.6 + .1[4 - .1 + 2(1.6)]$

$.1[3.9 + 3.2]$

$.1(7.1)$

$.71$

$y_2 = \boxed{2.31}$

$y_3 = 2.31 + .1[(4 - .2) + 2(2.31)]$

$3.8 + 4.62$

8.42

$y_3 = 2.31 + .842 = 3.152$

·on exam:

- slope fields 2.1, 2.2, 2.3
- Euler's Method approximation } 2.7

Exam { 1.1 – 1.3
2.1 – 2.3, 2.5, 2.7

Review Partial Fractions

Practice Exams

$\frac{dy}{dt} = ry\ln\left(\frac{K}{y}\right)$ cannot write as $y' + p(t)y = g(t)$

$\int \frac{dy}{y\ln\left(\frac{K}{y}\right)} = \int r\,dt$

Let $u = \ln\frac{y}{K}$

$\int \frac{1}{y\ln\left(\frac{K}{y}\right)}dy = rt + C \quad du = \frac{1}{\left(\frac{y}{K}\right)}$

7.2 Direction Fields and Euler's Method

It is impossible to solve most differential equations exactly, but we can still learn a lot about the solution through a graphical approach (direction fields) or a numerical approach (Euler's method)

Direction Fields

Suppose we are asked to sketch the graph of the solution of the initial-value problem

$$y' = x + y \qquad y(0) = 1$$

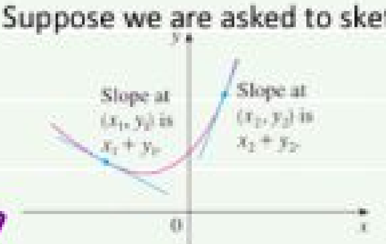


FIGURE 1
A solution of $y' = x + y$

Although we have no equation for the solution, we have been given a recipe for the slope at all any point (x, y). In particular, we glean that the slope at $(0,1)$ is equal to $0 + 1 = 1$

FIGURE 2
Beginning of the solution curve thro

By sketching short line segments at a number or points with slope $(x + y)$, we obtain a direction fields, which is helpful in interpolating what the solution graph should look like.

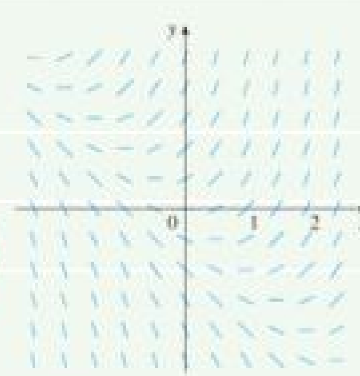


FIGURE 3
Direction field for $y' = x + y$

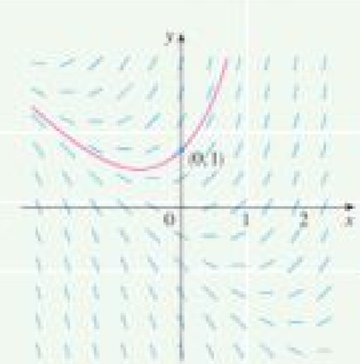
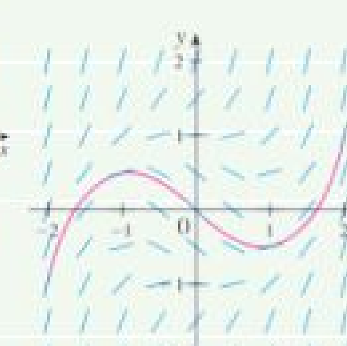


FIGURE 4
The solution curve through (0, 1)



Euler's Method

The idea here is to start at the point given by the initial value and proceed along the direction indicated by the direction field. After a short distance, look at the slope at the new location, and continue along that direction. Each stop is re-evaluation of what the slope should be based on our differential. By stopping more frequently (decreasing step size), this method yields successively more precise approximations.

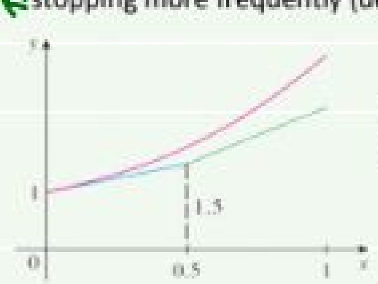


FIGURE 13
Euler approximation with step size 0.5

Euler's Method states that approximate values for the solution of the initial-value problem

$$y = F(x, y), \; y(x_0) = y_0,$$

with step size h, at $x_n = x_{n-1} + h$, are

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}) \qquad n = 1, 2, 3, \cdots$$

FIGURE 14
Euler approximation with step size 0.25

$y'' = f(t, y, y')$

$y'' = g(t) - p(t)y' - q(t)y$ solve for r_1, r_2 then solve for c_1 and c_2

$e^{rt} \quad re^{rt} \quad r^2e^{rt}$

$a[r^2e^{rt}] + b[re^{rt}] + ce^{rt} = 0$

$e^{rt}(ar^2 + br + c) = 0$

$ar^2 + br + c = 0$ is called the charactristic equation of the differential equation

solve for r_1 and r_2

$$= \frac{-b \pm [b^2 - 4ac]^{\frac{1}{2}}}{2a}$$

$[r^2 + 5r + 6]e^{rt} = 0$

$(r+2)(r+3) \quad c_1e^{-2t} + c_2e^{-3t} = y$

$r_1 = -2 \quad r_2 = -3$

$y' = -2c_1e^{-2t} - 3c_2e^{-3t}$

$y(0) = 2 \leftrightarrow c_1 + c_2 = 2$

$y'(0) = 3 = -2c_1 - 3c_2$

$y'(0) = 3$

$c_1 = 2 - c_2$

$c_2 = 2 - c_1$

$c_1 = 9$

$-2(2 - c_2) - 3c_2 = 3 \quad c_2 = -7$

$-4 + 2c_2 - 3c_2$

$-4 - \quad c_2 = 3$

$+4 \quad +4$

$\underline{y(t) = 9e^{-2t} - 7e^{-3t}}$

$\frac{3}{2}c_1 + \frac{1}{2}(2 - c_1) = \frac{1}{2}$

$4y'' - 8y' + 3y = 0 \quad \cancel{y(0) = 2}$

$y'(0) = \frac{1}{2}$

$y(0) = 2$

$y'(0) = \frac{1}{2}$

$\frac{3}{2}c_1 + 1 - \frac{c_1}{2} = \frac{1}{2}$

$-1 \quad -1$

$e^{rt}(4r^2 - 8r + 3)$

$4r^2 - 8r + 3$

$c_1e^{\frac{3}{2}t} + c_2e^{\frac{1}{2}t} = y \leftrightarrow y' = \frac{3}{2}c_1e^{\frac{3}{2}t} + \frac{1}{2}c_2e^{\frac{1}{2}t}$

$$\frac{8 \pm [64 - 4(4)(3)]^{(1/2)}}{2(4)} = \frac{8 \pm 4}{8} = \frac{12}{8}, \frac{4}{8} = \frac{3}{2}, \frac{1}{2}$$

$c_1 + c_2 = 2$

$2 - c_1 = c_2$

$c_1 = \frac{-1}{2}$

$c_2 = \frac{5}{2}$

$e^{i\mu t} = \cos\mu t + i\sin\mu t$

$y'' + y' + 9.25y = 0$

$(r^2 + r + 9.25)e^{rt}$

$r^2 + r + 9.25 = 0$

$\dfrac{-1 \pm [1^2 - 4(1)(9.25)]^{1/2}}{2(1)}$

$\dfrac{-1 + \sqrt{1 - 37}}{2}$

$\dfrac{-1 \pm 6i}{2}$

$\dfrac{-1 \pm \sqrt{-36}}{2}$

$-\frac{1}{2} + 3i, -\frac{1}{2} - 3i$

$-\frac{1}{2} \pm 3i$

$y = c_1 e^{-t/2}\cos(3t) + c_2 e^{-t/2}\sin(3t)$

$= e^{-t/2}(c_1\cos(3t) + c_2\sin(3t))$

$y(0) = 2 \quad y'(0) = 8$

$c_1 = 2 \quad y' =$

$y'' + 4y' + 4y = 0$

$e^{rt}(r^2 + 4r + 4) = 0$

$e^{rt} \neq 0 \rightarrow (r+2)(r+2)$

$r = -2 \quad y_1 = e^{-2t} \quad y_2 = te^{-2t}$

$c_1 e^{-2t} + c_2 te^{-2t} = y(t) \quad y(0) = 1$

$y'(0) = 3$

$c_1 = y_0 = 1$

$y' = -2c_1 e^{-2t} - c_2[-2te^{-2t} + e^{-2t}]$

$16y'' - 8y' + 145y = 0 \quad y(0) = -2 \quad y'(0) = 1$

$\dfrac{8 \pm [64 - 4(16)(145)]^{1/2}}{32}$

$\frac{8}{32} \pm \frac{96i}{32}$

$\frac{1}{4} \pm 3i$

$y'' + 25y = 0 \quad y(0) = 3 \quad y'(0) = 2$

$(r^2 + 25) = 0$

$r = \pm 5i$

$y(t) = c_1\cos 5t + c_2\sin 5t$

$c_1 = 3 \quad c_2 = \frac{2}{5}$

$y'(t) = -15\sin 5t + 5c_2\cos 5t \Rightarrow y'(0) = 2$

$y'' + 2y' + y = 0$

$(r^2 + 2r + 1)e^{rt} = 0$

$r^2 + 2r + 1 = 0 \quad y_1 = e^{-rt}$

$(r+1)(r+1)$

$r = -1 \quad y_2 = te^{-rt}$

$y_1 = e^{-t} \quad y_2 = te^{-t}$

μ = imaginary part
λ = real part.

$b^2 - 4ac < 0 \Rightarrow x = \lambda \pm \mu i$

$\lambda > 0 \to$ growing exponential envelope

$\lambda < 0 \to$ decaying exponential envelope

$\lambda = 0 \to$ constant oscillation amplitude

Look at double angle!

By definition, The Wronskian of the functions f and g is the determinant $\begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} = f(t)g'(t) - f'(t)g(t).$

$e^{2t} 4e^{4t} - 2e^{2t}e^{4t}$

$4e^{6t} - 2e^{6t} = 2e^{6t}$

$t(g'(t)) - 1(g(t)) = t^2 e^{2t}$

$\frac{d}{dt} \frac{te^{2t}}{2} \Rightarrow \frac{1}{2} \cdot e^{2t} + t \cdot e^{2t}$

$\frac{t}{2} e^{2t} + t^2 e^{2t} - 1\left(\frac{t}{2} e^{2t}\right)$ ✓

Laplace transform

$$L\{f(t)\} = F(s) = \int_{\alpha}^{\beta} e^{-st} f(t)$$

$$\int_{\alpha}^{\infty} f(t)\,\mathrm{d}t = \lim_{A \to \infty} \int_0^A e^{ct}\,\mathrm{d}t = \lim_{A \to \infty} \frac{1}{c}\left(e^{cA} - 1\right)$$

A mass that weighs 8 lbs stretches a spring 6 inches. The system is acted on by an external force of 8sin 8t. If the mass is pulled down 3 inches and then released, determine the position at all times t. Take g = 32 ft/s

$\to \vec{w} = mg \to \frac{w}{g} = m \to m = 8/32 \to 1/4$

$\to \vec{F_s} = -kL \to k = \frac{8}{1/2} = 16 \frac{lb}{ft} \to \frac{1}{4}u'' + 16u = 8\sin 8t \to u'' + 64u = 8\sin 8t$

$r^2 = -64$

$r = \pm 8i$

$\lambda = 0 \quad \mu = 8$

$y_g = c_1 \cos(8t) + \underbrace{c_2 \sin(8t)}_{\text{but } 8\sin 8t \text{ is a soln.}} \to y_p$

$\to$ Let $y_p = At\cos 8t + Bt\sin 8t$

$y_p' = A\cos 8t - 8At\sin 8t + B\sin 8t + 8Bt\cos 8t$

$y_p'' = -8A\sin 8t - 8A\sin 8t + 64At\cos 8t + 8B\cos 8t + 8B\cos 8t$ $- 64Bt\sin 8t$

$\frac{1}{4}\left(-16A\sin 8t + 64At\cos 8t + 16Bt\cos 8t - 64Bt\sin 8t\right)$

$-4A\sin 8t - 16At\cos 8t + 4B\cos 8t - 16Bt\sin 8t$

$+16(At\cos 8t + Bt\sin 8t)$

$\to 4B\cos 8t + 4A\cos 8t = 8\sin 8t$

$B = 2 \quad A = 0$

$y = c_1\cos 8t + c_2\sin 8t + 2t\sin 8t$

$y(0) = 3 = c_1, \; c_2 = 0$

$y'(0) = 0$

$y = 3\cos 8t + 2t\sin 8t$